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K_1 OF A CURVE OF GENUS ZERO(1)

BY

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ABSTRACT. We determine the structure of the vector bundles on a curve of genus zero and calculate the "universal determinant" K_1 of such a curve.

1. Introduction. Let F be a field. Then there is a bijection between (nonsingular projective irreducible) curves of genus zero over F and central simple algebras of rank 4 over F. If X is such a curve then X is isomorphic to a plane curve of degree 2, and there is a separable extension $[K: F]$ of degree 2 such that $X \times_F K \cong P^1_K$, the projective line over K.

Let $\mathcal{V}$ be the category of vector bundles (= locally free sheaves of finite type) on X and $\mathfrak{M}$ the (abelian) category of coherent sheaves on X. Let K_1 be the "universal determinant" K_1 as defined in [3, Chapter VIII]. The groups $K_1(\mathcal{V})$ and $K_1(\mathfrak{M})$ are both defined. Set $K_1(X) = K_1(\mathcal{V})$. In this paper we prove that if X is the curve of genus zero over F corresponding to the central simple algebra A then $K_1(X) \cong K_1(F) \oplus K_1(A)$. At the end of the paper it is proved that the inclusion of categories $\mathcal{V} \rightarrow \mathfrak{M}$ induces an isomorphism $K_1(\mathcal{V}) \rightarrow K_1(\mathfrak{M})$ so $K_1(X)$ could have been defined with coherent sheaves instead of vector bundles. If A is the ring of 2×2 matrices over F, then $X = P^1_F$ and the formula reads $K_1(X) \cong K_1(F) \oplus K_1(A) = F^* \oplus F^*$ (where F^* denotes the nonzero elements of F). This has already been proved in [8] or [9] (working with coherent sheaves in the first case and vector bundles in the second) so we can confine ourselves to the case where A does not split, i.e. is a division ring of rank 4.

Recently [7] Quillen has developed a theory of higher K's for schemes, and in [7] he calculates the K-theory for Severi-Brauer schemes, the simplest example of which are the curves of genus zero. The result that I have obtained agrees with his, although Gersten in [6] has proved that if X is a nonsingular elliptic curve over $\mathbf{C}$ Quillen's K_1 is not the same as the "universal determinant" K_1.

This paper is based on the second half of [11]. Throughout $\mathbf{Z}$ denotes the integers, $\mathbf{R}$ the real numbers, and $\mathbf{C}$ the complex numbers.

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2. The structure of vector bundles on X. The results in this section seem to be more or less "known" (see the discussion in [7, §8]). I include them here because I need the detailed description of the vector bundles for my calculation of K_1 and do not know a suitable reference.

Let X be a curve of genus 0 over the field F, $X \not\cong P^1_F$, and let K be a separable extension of degree 2 such that $X_K = X \times_F K \cong P^1_K$. If $f: X_K \to X$ is the morphism obtained from the change of field, then f^* is an injection on isomorphism classes of vector bundles or coherent sheaves [12, remark at the end of §1]. We use this, together with the known structure of vector bundles on P^1_K to determine the structure of vector bundles on X.

The Krull-Schmidt theorem holds for vector bundles on X and X_K because all Hom's are finite dimensional vector spaces over the ground field [2]. That is, every vector bundle can be written in a unique manner as the direct sum of indecomposable vector bundles. If we write $P^1_K = \operatorname{Proj} K[T_0, T_1]$ and let $\mathcal{O}(1)$ be the canonical line bundle determined by this projective structure, then the indecomposable vector bundles on X_K are just the line bundles $\mathcal{O}(n)$. Some discussion of this can be found in [10]. Furthermore $\Gamma(\mathcal{O}(n))$ is a vector space of dimension $n+1$ over K if $n \geq 0$ and is zero otherwise. Also $\operatorname{Hom}(\mathcal{O}(m), \mathcal{O}(n)) = \Gamma(\mathcal{O}(-m)\otimes\mathcal{O}(n)) = \Gamma(\mathcal{O}(n-m))$. Therefore there are no nonzero morphisms from $\mathcal{O}(m)$ to $\mathcal{O}(n)$ unless $n \geq m$.

The Picard group of X_K is $\mathbf{Z}$, generated by $\mathcal{O}(1)$. The Picard group of X is $\mathbf{Z}$ also, generated by $\mathcal{O}(1)$, where $\mathcal{O}(1)$ is defined by the projective structure of X as a second degree curve. Also $f^*\mathcal{O}(1) = \mathcal{O}(2)$. (I will write simply $\mathcal{O}(n)$, it being clear from the context whether this is a bundle on X or X_K.)

The following lemma will be used throughout this discussion:

Lemma. *Let Y be a scheme of finite type over a noetherian ring A. Let B be a flat A-algebra, and let U be an open subset of Y. Write $Y' = Y \otimes_A B$ and let $f: Y' \to Y$ be the morphism induced by change of base. Let N be a quasicoherent sheaf on Y and $N' = f^*(N)$. Then $\Gamma(f^{-1}(U), N') = B \otimes_A \Gamma(U, N)$.*

Proof. First of all the lemma is true if U is affine, by the construction of product in the category of schemes and the behavior of f^* in the affine case. If U is not affine then U can be covered by a finite number of affine schemes U_i $(1 \leq i \leq n)$. If we write $U_{ij} = U_i \cap U_j$ then U_{ij} will be affine also. We have an exact sequence

$$0 \to \Gamma(U, N) \to \Pi_i \Gamma(U_i, N) \rightrightarrows \Pi_{i,j} \Gamma(U_{ij}, N)$$

since N is a sheaf. Tensoring with B and using the lemma for affine sets we get an exact sequence

$$0 \to B \otimes_A \Gamma(U, N) \to \Pi_i \Gamma(f^{-1}(U_i), N') \rightrightarrows \Pi_{i,j} \Gamma(f^{-1}(U_{ij}), N').$$

But $f^{-1}(U)$ is covered by the $f^{-1}(U_i)$ and $f^{-1}(U_{ij}) = f^{-1}(U_i) \cap f^{-1}(U_j)$. Therefore we have an exact sequence

$$0 \to \Gamma(f^{-1}(U), N') \to \Pi_i \Gamma(f^{-1}(U_i), N') \rightrightarrows \Pi_{i,j} \Gamma(f^{-1}(U_{ij}), N').$$

Therefore $B \otimes_A \Gamma(U, N) = \Gamma(f^{-1}(U), N')$ as required.

In the application $A = F$, $B = K$ and $Y = U = X$. The lemma is false if B is not a flat A-module. For example, let $A = K[T_1, T_2]$, $Y = \operatorname{Spec} A$, $U = \operatorname{Spec} A -$ (origin) and $B = K$, the homorphism $A \to B$ given by sending T_1 and T_2 to 0. If N is the structure sheaf of Y, then $\Gamma(U, N) = A$, and $B \otimes_A \Gamma(U, N) = B$. But $f^{-1}(U) = \emptyset$, so $\Gamma(f^{-1}(U), N') = 0$. (This example was pointed out to me by Paul-Jean Cahen.)

The structure of the vector bundles on X is given by the following theorem:

Theorem 1. *Let X be a curve of genus 0 over the field F, $X \not\cong P^1_F$, and let K be a separable extension of degree 2 such that $X_K = X \times_F K \cong P^1_K$. Let $f: X_K \to X$ be the morphism obtained from change of base. Then the vector bundle $E(n) = f_* \mathcal{O}(n)$ (n odd) on X is indecomposable of rank 2. Every vector bundle on X can be written uniquely (up to order of summands) as the direct sum of line bundles $\mathcal{O}(n)$, and the bundles $E(n)$.*

Proof. First we show that $E(n)$ is indecomposable. If $E(n)$ is decomposable, then $E(n) = \mathcal{O}(n_1) \oplus \mathcal{O}(n_2)$ and $f^*E(n) = \mathcal{O}(2n_1) \oplus \mathcal{O}(2n_2)$. The Galois group $\mathbf{Z}/2\mathbf{Z}$ of K over F acts on the vector bundles on X_K in an obvious way (denoted by a $\bar{\ }$). By (2′) of [12] we have $f^* f_* \mathcal{O}(n) = \mathcal{O}(n) \oplus \overline{\mathcal{O}(n)}$. The equation $f^*\mathcal{O}(1) = \mathcal{O}(2)$ proves that the Galois group acts trivially. Therefore we must have $f^*(E(n)) = \mathcal{O}(n) \oplus \mathcal{O}(n)$. Hence $E(n)$ must be indecomposible, otherwise the Krull-Schmidt theorem would be violated.

If n is even, $f_* \mathcal{O}(n) = \mathcal{O}(n/2) \oplus \mathcal{O}(n/2)$. For we get $\mathcal{O}(n) \oplus \mathcal{O}(n)$ on both sides if we apply f^*, and f^* is an injection on isomorphism classes. Now suppose that V is a vector bundle on X. Then $f^*(V) = \bigoplus_i \mathcal{O}(n_i)$, so $f_* f^*(V) = V \oplus V$ is the direct sum of the $\mathcal{O}(n_i)$ and $E(n_i)$. By the Krull-Schmidt theorem so also is V. This completes the proof of Theorem 1.

I will conclude this section by making some general remarks about the vector bundles on X. Let Hom_F denote morphisms of vector bundles on X, and Hom_K denote morphisms of vector bundles on X_K. First of all, $\operatorname{Hom}_F(\mathcal{O}(n), \mathcal{O}(m)) = 0$ if $n > m$ and is nonzero if $n \leq m$. Also $\operatorname{Hom}_F(\mathcal{O}(n), E(m)) \otimes_F K =$

$\mathrm{Hom}_K\,(f^*\mathcal{O}(n), f^*E(m)) = \mathrm{Hom}_K\,(\mathcal{O}(2n), \mathcal{O}(m) \oplus \mathcal{O}(m))$ so $\mathrm{Hom}_F\,(\mathcal{O}(n), E(m)) = 0$ if $2n > m$ and is nonzero if $2n < m$. By applying f^* and using the fact that f^* is an injection on isomorphism classes we can prove that $E(n) \otimes \mathcal{O}(m) \cong E(n+2m)$, $E(n)^* = E(-n)$ (*denotes dual) and $E(n) \otimes E(m) \cong 4\,\mathcal{O}((m+n)/2)$. From the last isomorphism it follows that $\mathbf{Hom}\,(E(n), E(m)) \cong E(-n) \otimes E(m) \cong 4\,\mathcal{O}((m-n)/2)$. Thus $\mathrm{Hom}_F\,(E(n), E(m)) = 0$ if $n > m$ and is nonzero if $n \leq m$. Hence we may linearly order the vector bundles $\cdots\ E(-3), \mathcal{O}(-1), E(-1), \mathcal{O}, E(1), \mathcal{O}(1), E(3), \mathcal{O}(2), \cdots$ with nonzero morphisms going only to the right. One can also show that $\Lambda^2 E(1) = \mathcal{O}(1)$ by applying f^* to both sides.

Now we consider $\mathrm{Hom}_F\,(E(n), E(n))$. From the above it is a 4 dimensional vector space over F, and since $E(n)$ is indecomposable, there are no nontrivial idempotents. Finally $\mathrm{Hom}_F\,(E(n), E(n)) \otimes_F K$ is the ring of 2×2 matrices over K. Thus $\mathrm{Hom}_F\,(E(n), E(n))$ is semisimple and therefore a division ring over F. The $\mathrm{Hom}_F\,(E(n), E(n))$ are all isomorphic, since $E(n) \otimes \mathcal{O}(m) \cong E(n+2m)$.

More precisely, if F is a field of characteristic $\neq 2$ then the equation for for the plane curve X (in homogeneous co-ordinates) is (for suitable choice of variables) $T_0^2 - aT_1^2 - bT_2^2 = 0$, $a, b \in F$ and $\mathrm{Hom}_F\,(E(-1), E(-1))$ is isomorphic to the quaternion algebra (a, b) (as defined on p. 96 of [13]). If the characteristic F is 2, then X is given by the equation $aT_1^2 + T_1T_2 + bT_2^2 + cT_0^2 = 0$ with $a, b, c \in F$, and $\mathrm{Hom}_F\,(E(-1), E(-1))$ is isomorphic to the Clifford algebra of the quadratic form $acv^2 + cuv + bcv^2$ as defined in [1, p. 150]. The characteristic $\neq 2$ case was proved by a straightforward but tedious calculation in [11] and the characteristic 2 case can be proved in a similar manner. I will omit these proofs because all we need to know for the calculation in §3 is that $\mathrm{Hom}_F\,(E(-1), E(-1))$ is a division ring of dimension 4 over its centre F. In fact, $\mathrm{Hom}_F\,(E(-1), E(-1))$ is just the central simple algebra corresponding to X in the bijection mentioned at the beginning of the paper.

3. **Calculation of K_1.** We now calculate the group $K_1(X)$, where X is as in §2. Let V be a vector bundle on X, with automorphism α. Let $V = n_1V_1 \oplus n_2V_2 \oplus \cdots \oplus n_rV_r$ be an expression for V as the direct sum of indecomposable vector bundles V_i which are ordered so that there exist nonzero morphisms $V_i \to V_j$ if and only if $i \leq j$. Using this direct sum decomposition α can be represented by a lower triangular matrix, with r $n_i \times n_i$ blocks α_i down the diagonal having entries in either F or A depending on whether V_i is of rank 1 or 2. Here $A = \mathrm{Hom}_F\,(E(-1), E(-1))$, which is the quaternion algebra (a, b) that determines the curve if the characteristic $\neq 2$, or a certain Clifford algebra is characteristic $= 2$. In both cases A is a division ring. The α_i are invertible. One of the defining relations of K_1 is that if we have a short exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ in $\mathcal{V}$ and $\beta_1, \beta_2, \beta_3$ are automorphisms such that

$$\begin{array}{ccccccccc} 0 & \to & V_1 & \to & V_2 & \to & V_3 & \to & 0 \\ & & \downarrow{\beta_1} & & \downarrow{\beta_2} & & \downarrow{\beta_3} & & \\ 0 & \to & V_1 & \to & V_2 & \to & V_3 & \to & 0 \end{array}$$

is commutative, then $\kappa_1(V_2, \beta_2) = \kappa_1(V_1, \beta_1) + \kappa_1(V_3, \beta_3)$, where κ_1 denotes the canonical image in K_1. Repeated application of this proves that $\kappa_1(V, \alpha) = \Sigma_{i=1}^{r} \kappa_1(n_i V_i, \alpha_i)$. Write A^* and F^* for the nonzero elements of A and F respectively. One sees easily that $\kappa_1(n_i V_i, \alpha_i) = \kappa_1(V_i, a_i)$ for some $a_i \in A^*$ or F^* (depending on whether rank $V_i = 2$ or 1). Thus we have found generators for $K_1(X)$. We now try to reduce the number of generators by using exact sequences.

First of all, there is an exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0$ on X since $\mathcal{O}(1)$ is generated by two global sections. (The kernel of the resulting map $\mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0$ is a line bundle and must be $\mathcal{O}(-1)$ because the degree is additive.) Tensoring with $\mathcal{O}(n)$ we get exact sequences of the form $0 \to \mathcal{O}(n-1) \to \mathcal{O}(n) \oplus \mathcal{O}(n) \to \mathcal{O}(n+1) \to 0$ and these enable us to replace all the generators of the form $\kappa_1(\mathcal{O}(n), \lambda)$ by those of the form $\kappa_1(\mathcal{O}, \lambda)$ and $\kappa_1(\mathcal{O}(1), \lambda)$, $\lambda \in F^*$. Tensoring with $E(n)$ yields exact sequences of the form $0 \to E(n-2) \to E(n) \oplus E(n) \to E(n+2) \to 0$ and these enable us to replace the generators $\kappa_1(E(n), \mu)$, $\mu \in A^*$, by those of the form $\kappa_1(E(-1), \mu)$ and $\kappa_1(E(1), \mu)$. There is a nonzero morphism $E(1) \to \mathcal{O}(1)$ which must be onto, otherwise the image would be isomorphic to $\mathcal{O}(n)$ for some $n \leq 0$ and there are no nonzero maps $E(1) \to \mathcal{O}(n)$, $n \leq 0$. The kernel is a line bundle, which must be isomorphic to $\mathcal{O}$ since we have seen that $\Lambda^2 E(1) \cong \mathcal{O}(1)$. Therefore we have an exact sequence $0 \to \mathcal{O} \to E(1) \to \mathcal{O}(1) \to 0$. This enables us to get rid of the generators of the form $\kappa_1(\mathcal{O}(1), \lambda)$, $\lambda \in F^*$. Finally, on X_K we have an exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0$. If we apply f_* we get an exact sequence $0 \to E(-1) \to \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \to E(1) \to 0$. The map $\mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0$ on X_K is onto on global sections. Therefore so also is the map $4\,\mathcal{O} \to E(1)$ on X_F (since the global sections remain the same). If $\mu \in \operatorname{Aut} E(1) = A^*$ then μ induces an automorphism of $\Gamma E(1)$ which is a 4 dimensional vector space over F (as can be seen by applying f^*). Therefore μ can be lifted to an automorphism of $4\mathcal{O}$, and hence to an automorphism of the whole exact sequence $0 \to E(-1) \to 4\mathcal{O} \to E(1) \to 0$. By taking duals any automorphism of $E(-1)$ also extends to an automorphism of the sequence. The generators of $K_1(X)$ are finally reduced to elements of the form $\kappa_1(\mathcal{O}, \lambda)$ and $\kappa_1(E(1), \mu)$ for $\lambda \in F^*$, $\mu \in A^*$. This can be rephrased by saying there is a surjection ϕ: $F^* \oplus H^* \to K_1(X)$ defined by $\phi(\lambda) = \kappa_1(\mathcal{O}, \lambda)$ ($\lambda \in F^*$) and $\phi(\mu) = \kappa_1(E(1), \mu)$ ($\mu \in H^*$).

Now consider the reduced norm $N: A^* \to F^*$. It is proved in [14, Corollary p. 334], that the kernel of N is the commutator subgroup of A^* (the index being 2 which is square free). Let NA^* denote the image of N. The abelianized group of A^* is therefore NA^*. Therefore ϕ induces a surjection (also denoted ϕ) $\phi: F^* \oplus NA^* \to K_1(X)$.

We now have homomorphisms det: $K_1(X) \to F^*$ and $\chi: K_1(X) \to F^*$. The map det is defined by taking exterior powers. If V is a vector bundle of rank r, then $\alpha \in \operatorname{Aut} V$ induces an automorphism $\det \alpha$ of $\Lambda^r V$. But $\Lambda^r V$ is a line bundle so $\operatorname{Aut} \Lambda^r V \cong F^*$ (canonically). Then $\det \kappa_1(V, \alpha) = \det \alpha$. The vector spaces $H^0(X, V)$ and $H^1(X, V)$ are finite dimensional, and $\alpha \in \operatorname{Aut} V$ induces automorphisms of these vector spaces. These automorphisms will be denoted α_0 and α_1 respectively. Then $\chi(V, \alpha) = (\det \alpha_0)(\det \alpha_1)^{-1}$. If $0 \to (V_1, \alpha_1) \to (V, \alpha) \to (V_2, \alpha_2) \to 0$ is exact, then $\chi(V, \alpha) = \chi(V_1, \alpha_1)\chi(V_2, \alpha_2)$ by the exact sequence of cohomology. That $\chi(V, \alpha\beta) = \chi(V, \alpha)\chi(V, \beta)$ is obvious, so χ defines a homomorphism $\chi: K_1(X) \to F^*$.

We now examine what these homomorphisms do to the generators of $K_1(X)$. It is clear that $\det(\mathcal{O}, \lambda) = \lambda$, $\lambda \in F^*$. End $E(1) = A$ is a subalgebra of $\operatorname{End} E(1) \otimes_F K = \operatorname{End}_K (\mathcal{O}(1) \oplus \mathcal{O}(1))$ which is the ring of 2×2 matrices over K. Det commutes with base change. Therefore by the definition of reduced norm [5, p. 142], we have that $\det(E(1), \mu) = N\mu (\mu \in A^*)$. The Riemann-Roch theorem says that $\dim_F H^0(X, V) - \dim_F H^1(X, V) = \text{degree}(\Lambda^r V) + r$, where rank $V = r$. If we take $V = \mathcal{O}$, then degree $\mathcal{O} = 0$, $r = 1$, so we get $\dim H^1(X, \mathcal{O}) = 0$. Therefore $\chi(\mathcal{O}, \lambda) = \lambda$ also. If we take $V = E(1)$, then $\dim H^0(X, E(1)) = 4$, $\text{degree}(\Lambda^2 E(1)) = \text{degree}\, \mathcal{O}(1) = 2$, and $r = 2$. Therefore $\dim H^1(X, E(1)) = 0$. Thus $\chi(E(1), \mu) = \det \mu_0$. But $H^0(X, E(1)) = \Gamma(E(1))$ is a one dimensional vector space over A, so $\det \mu_0$ is the usual norm, which is the square of the reduced norm. That is, $\det \mu_0 = (N\mu)^2$.

Now define a homomorphism $\psi: K_1(X) \to F^* \oplus F^*$ by $\psi = ((\det)^2\chi^{-1}, \chi(\det)^{-1})$. Then $\psi\kappa_1(\mathcal{O}, \lambda) = (\lambda, 1)$ and $\psi\kappa_1(E(1), \mu) = (1, N\mu)$. Therefore the image of ψ is $F^* \oplus NA^*$ and $\psi\phi: F^* \oplus NA^* \to F^* \oplus NA^*$ is the identity. We have already seen that ϕ is onto. Therefore ϕ is an isomorphism. This proves

Theorem 2. *Let F be a field and let X be a nonsingular curve of genus 0 which is not isomorphic to P^1_F. Let the division algebra A be the endomorphism ring of the indecomposable vector bundle $E(1)$ of rank 2 on X. Let NA^* denote the image of the reduced norm $N: A^* \to F^*$. Then there is an isomorphism $\phi: F^* \oplus NA^* \to K_1(X)$. (Note that $F^* = K_1(F)$ and $NA^* = K_1(A)$.)*

4. **Further remarks.** We first consider the homomorphism $\Phi: K_0(X) \otimes_Z F^* \to K_1(X)$ defined by $\Phi\kappa_0(V) \otimes \lambda = \kappa_1(V, \lambda)$. Here K_0 denotes the Grothendieck

group of vector bundles with relations coming from short exact sequences, and $\kappa_0(V)$ denotes the image of V in K_0. Then $K_0(X) = \mathbf{Z} \oplus \mathbf{Z}$ with the second copy of $\mathbf{Z}$ being the Picard group. If we use this to identify $K_0(X) \otimes_{\mathbf{Z}} F^*$ with $F^* \oplus F^*$, then $\psi\Phi(\lambda, \mu) = \psi\kappa_1(\mathcal{O}, \lambda) + \psi\kappa_1(\mathcal{O}(1), \mu) = (\lambda, 1) + (\mu^{-1}, \mu^2) = (\lambda\mu^{-1}, \mu^2)$. If $F = \mathbf{R}$, since ψ is an isomorphism, we see that Φ is onto but has nontrivial kernel $(-1, -1)$. We note that in general NA^* will be bigger than $(F^*)^2$. If this is the case $\psi\Phi$ will not be onto, so neither is Φ.

In [8] it was proved that Φ is an isomorphism for X a projective nonsingular variety over an algebraically closed field.

We now consider the homomorphism $f^*: K_1(X) \to K_1(X_K)$ induced by the change of base. $K_1(X_K) = K^* \oplus K^*$ generated by $\kappa_1(\mathcal{O}, \lambda)$ and $\kappa_1(\mathcal{O}(1), \lambda)$, $\lambda \in K^*$. The action of the Galois group $G = \mathbf{Z}/2\mathbf{Z}$ of K over F on $K_1(X_K)$ is just the obvious action on each copy of K^*. The image of f^* is contained in $K_1(X_K)^G$ (the fixed subgroup under the action of G) and by §2 of [12], the kernel and cokernel of the map $f^*: K_1(X) \to K_1(X_K)^G = F^* \oplus F^*$ are both killed by 2. One can check (using the above identifications of $K_1(X_K)$ with $K^* \oplus K^*$ and $K_1(X)$ with $F^* \oplus NA^*$) that $f^*(\lambda, \mu) = (\lambda, \mu)$, $\lambda \in F^*$, $\mu \in NA^*$. Therefore f^* is injective, and the cokernel of $f^*: K_1(X) \to K_1(X_K)^G$ is F^*/NA^* which is indeed killed by 2 because every square is a reduced norm.

We conclude by proving that coherent sheaves and vector bundles give the same K_1.

Theorem 3. *Let Y be a regular projective scheme of finite type over a field F. Let $\mathcal{V}$ be the category of vector bundles and $\mathfrak{M}$ the category of coherent sheaves on Y. Then the homomorphism $K_1(\mathcal{V}) \to K_1(\mathfrak{M})$ induced by the inclusion of categories is an isomorphism.*

Proof. This follows from Theorem 5, p. 72 of [4]. The hypotheses are all immediate except (c). Let N be an object in $\mathfrak{M}$. If n is sufficiently large then $N \otimes \mathcal{O}(n)$ will be generated by its global sections (which form a finite dimensional vector space over F since Y is projective). If α is an endomorphism of N, then $\alpha \otimes 1$ induces an endomorphism of the global sections of $N \otimes \mathcal{O}(n)$. Suppose $\dim \Gamma(N \otimes \mathcal{O}(n)) = m$. Choose a basis for $\Gamma(N \otimes \mathcal{O}(n))$. We have a surjection $m\mathcal{O}_Y \to N \otimes \mathcal{O}(n) \to 0$ by mapping the unit sections of the copies of $\mathcal{O}_Y$ to the corresponding basis vectors for $\Gamma(N \otimes \mathcal{O}(n))$. If $\Gamma(\alpha \otimes 1)$ has matrix A, then the endomorphism of $m\mathcal{O}$ given by the same matrix lifts $\alpha \otimes 1$. Tensoring with $\mathcal{O}(-n)$ we see that α can be lifted to an endomorphism of $m\mathcal{O}(-n)$, which proves (c). Theorem 3 now follows.

If X were affine a similar result holds by [4, Theorem 3], but if Y is neither affine nor projective then I do not know if the corresponding result holds. If

$A_F^2 = \mathrm{Spec}\ F[T_0, T_1]$ (the affine plane) and $Y = A_F^2 -$ (origin) then I suspect that $\kappa_1(H, \lambda)$ where H the structure sheaf of Spec $F[T_0]$ restricted to Y and λ is multiplication by T_0 does not lie in the image of the homomorphism $K_1(\mathfrak{C}) \to K_1(\mathfrak{M})$ but I do not know how to prove it.

REFERENCES

1. C. Arf, *Untersuchungen über quadratische Formen in Körpern der Charakteristic* 2. I, J. Reine Angew. Math. 183 (1941), 148–167. MR 4, 237.

2. M. F. Atiyah, *On the Krull-Schmidt Theorem with application to sheaves*, Bull. Soc. Math. France 84 (1956), 307–317. MR 19, 172.

3. H. Bass, *Algebraic K-theory*, Benjamin, New York, 1968. MR 40 #2736.

4. H. Bass, A. Heller and R. Swan, *The Whitehead group of a polynomial extension*, Inst. Hautes Études Sci. Publ. Math. No. 22 (1964), 61–79. MR 30 #4806.

5. N. Bourbaki, *Eléments de mathématique*. XXIII. Part. 1. *Les structures fondamentales de l'analyse*. Livre II: *Algèbre*. Chap. 8: *Modules et anneaux semi-simples*, Actualités Sci. Indust., no. 1261, Hermann, Paris, 1958. MR 20 #4576.

6. S. Gersten, *Higher K-theory of rings*, Lecture Notes in Math., vol. 341, Springer-Verlag, Berlin, 1973, pp. 3–42.

7. D. Quillen, *Higher algebraic K-theory*. I, Lecture Notes in Math., vol. 341, Springer-Verlag, Berlin, 1973, pp. 85–147.

8. L. G. Roberts, *K_1 of some abelian categories*, Trans. Amer. Math. Soc. 138 (1969), 377–382. MR 40 #4340.

9. ———, *K_1 of projective r-space*, Proc. Amer. Math. Soc. 26 (1970), 587–592. MR 43 #6221.

10. ———, *Indecomposable vector bundles on the projective line*, Canad. J. Math. 24 (1972), 149–154. MR 45 #6828.

11. ———, *Real quadrics and K_1 of a curve of genus zero*, Mathematical Preprint No. 1971–60, Queen's University at Kingston, Ont.

12. ———, *Base change for K_0 of algebraic varieties*, Lecture Notes in Math., vol. 342, Springer-Verlag, Berlin, 1973, pp. 122–134.

13. W. Scharlau, *Quadratic forms*, Queen's Papers on Pure and Applied Mathematics, No. 22., Queen's University, Kingston, Ontario, 1969. MR 42 #4574.

14. S. Wang, *On the commutator group of a simple algebra*, Amer. J. Math. 72 (1950), 323–334. MR 11, 577.

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